Continuum-Based Crowd Flow Modeling

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1. The Importance of Crowd Flow Modeling

For our project, we were interested in learning about how crowds can be modeled using Partial Differential Equations (PDEs). Specifically, we chose to study a type of modeling known as Continuum-Based Crowd Flow Modeling.

Crowd flow modeling is an interesting and important field of modeling because crowd flow models have the potential to provide useful information about how a crowd will behave in a given situation or scenario without having to carry out an actual experiment with human test subjects. This information can then be used by the relevant authorities to help them develop strategies to control or manage a crowd more effectively.

One of the most famous examples of how crowd flow modeling can be applied to real-world problems is the use of crowd flow modeling to redesign the Jamarat bridge near Mecca in Saudi Arabia. The Jamarat bridge is a bridge used by millions of pilgrims during the last day of Haji. Prior to the bridge's redesign, there were 3 major pedestrian disasters due to poor crowd management and bridge design. By applying crowd flow modeling to study the effects of different bridge designs on the flow of crowds, the bridge is now much safer and the authorities are now able to use that crowd flow model to inform their placement of crowd control equipment and personnel.

Over the course of our project, we read the original paper by Hughes (2002)[1] who described a continuum-based approach to deriving the necessary PDEs needed to model the flow of pedestrians. We then read an additional paper by Huang et al (2009)[2], who revisited Hughes' continuum model for pedestrian flow and further refined both the equations that modeled crowd flow as well as the solution method to the PDEs. Using information from both papers, we simulated a 1-dimensional crowd flow situation using Matlab.

[1]Hughes, R. L. (2002). A continuum theory for the flow of pedestrians. *Transportation Research Part B, 36, 507 - 535*

[2]Huang, L., Wong, S. C., Zhang, M., Shu, C., Lam, W. H. K. (2009) Revisiting Hughes' dynamic continuum model for pedestrian flow and the development of an efficient solution algorithm. *Transportation Research Part B, 43, 127-141*

2. The Presentation

During our presentation, we will do the following:

1. Briefly discuss how to derive the PDEs that govern crowd flow through a space.
2. Highlight which equations the equations that are the result of modeling decisions
3. Briefly discuss the numerical method we used to solve the PDEs
4. Demonstrate the simple application of these equations through a numerical solution of a 1-dimensional case of crowd flow.

From our presentation, we hope you will get a sense for the general idea of how to develop the PDEs to model a crowd based on an understanding you might have about the behavior of the crowd you are trying to model. In addition, by presenting a simple example of how to numerically solve the PDEs, we hope you will have a basic understanding of how to apply the solution technique to your own set of PDEs should you encounter crowd flow PDEs in the future.

3. A Brief Overview of the Crowd Flow Model

In continuum-based crowd flow modeling, the essential idea is that, for relatively large crowds, larger-scale motions of the crowd are more important than individual differences in motion. As a result, the crowd is studied in terms of crowd density rather than motions of individuals.

In the past, crowds were routinely treated like fluids and fluid flow equations were used to model crowds. Now, although some of the principles are applied, the fluid flow equations themselves are rarely used.

Along this train of thought, we might begin by modeling a crowd in terms of a conservation law. Before we really launch into developing a model for crowd flow, the table below summarizes the variables we will be using throughout this discussion:

Table of Variables		
Variable	Description	Units
ρ	density	# people / m^2
u, v	velocity in x, y direction	m / s
$f(\rho)$	speed	m / s
$\hat{\phi}_x, \hat{\phi}_y$	directional cosines	
ϕ	potential	

Tabela 1. Table of the variables that will be used throughout this discussion

At this point, we should note that we will only summarize the equations needed to model crowd flow and a more developed explaination is provided in section 7 at the end of this handout.

3.1. The Mass Conservation Law

Coming back to the model derivation, the simplest model form that makes the most sense would be some sort of mass conservation law like this one:

$$\frac{\delta \rho}{\delta t} + \frac{\delta}{\delta x}(\rho u) + \frac{\delta}{\delta y}(\rho v) = 0 \tag{1}$$

In essence, Equation 1 applies the concept of mass consevation but is really a statement of "people conservaion". Physically, Equation 1 states that the change in density over time is the result of a net flow of people into or out of an area.

3.2. Capturing Crowd Behavior

We can then capture the behavior of the crowd through proper modeling of the crowd's velocity components u and v. We can therefore express the velocity components of the movement of the crowd as:

$$u = f(\rho)\hat{\phi_x} \tag{2}$$

$$v = f(\rho)\hat{\phi_y} \tag{3}$$

In Equations 2 and 3, the "behavioral characerisics of the crowd"are captured in the $\hat{\phi_x}$ and $\hat{\phi_y}$ terms. Those terms are commonly referred to as the direction cosines of the motion or the direction in which the pedestrians are walking.

We then introduce a quantity called potential, which is essentially a measure of how much a crowd wishes to get from their current position to their destination. Logically, we could therefore assume that a crowd will move in a direction that minimizes their potential and fulfills their wishes a fast as possible. Based on what we know about the gradient of a scalar field, we would expect the crowd to move ina direction that is perpendicular to lines of constant poential. Mathematically, we can express this as:

$$\hat{\phi_x} = \frac{-\frac{\delta \phi}{\delta x}}{\sqrt{(\frac{\delta \phi}{\delta x})^2 + (\frac{\delta \phi}{\delta y})^2}} \tag{4}$$

$$\hat{\phi_y} = \frac{-\frac{\delta \phi}{\delta y}}{\sqrt{(\frac{\delta \phi}{\delta x})^2 + (\frac{\delta \phi}{\delta y})^2}} \tag{5}$$

where ϕ is the potential.

At this point, the sources we read also recommend implementing one other equation to capture the fact that pedestrians will adjust their behavior to minimize the product of their travel time and a function of the density to avoid high density areas. Mathematically, this can be expressed as:

$$\frac{1}{\sqrt{(\frac{\delta \phi}{\delta x})^2 + (\frac{\delta \phi}{\delta y})^2}} = g(\rho)\sqrt{u^2 + v^2} \tag{6}$$

3.3. Putting It All Together

Now that we have Equations 1, 2, 3, 4 and 5, we can combine them together to obtain the first PDE that governs pedestrian flow:

$$-\frac{\delta\rho}{\delta t}+\frac{\delta}{\delta x}(\rho g(\rho)f^2(\rho)\frac{\delta\phi}{\delta x})+\frac{\delta}{\delta y}(\rho g(\rho)f^2(\rho)\frac{\delta\phi}{\delta y})=0 \tag{7}$$

We also need one additional equation to relate the potential, ϕ to ρ and that equation is Equation 6. Since $f(\rho)=\sqrt{u^2+v^2}$:

$$g(\rho)f(\rho)=\frac{1}{\sqrt{(\frac{\delta\phi}{\delta x})^2+(\frac{\delta\phi}{\delta y})^2}} \tag{8}$$

4. Modeling the 1-Dimensional Case

Using Equations 7 and 8, we developed a numerical solution for the one dimensional case of crowd flow in Matlab. To do this, we follow the recommendation made by Hughes (2002) and allow $g(\rho)$ to be equal to 1. We also assume the walking speed of the pedestrians, $f(\rho)$ can be modeled by a linear model:

$$f(\rho)=A-B\rho \tag{9}$$

where A is the free (i.e. when density is zero) walking speed of pedestrians and B controls the rate at which density affects the walking speed of pedestrains.

So, by applying Equation 9 as well as the above assumptions and evaluating Equation 7 and Equation 8, the following model is obained:

$$\rho_t=(A-2B\rho)\rho_x \tag{10}$$

Now that we have Equation 10, we can now solve the PDE numerically. If we use i to index our space unit and n to index our time unit, the following equation is obtained:

$$\frac{\rho_i^{n+1}-\rho_i^n}{\delta t}=\frac{\rho_i^n-\rho_{i-1}^n}{\delta x}(A-B\rho_i^n) \tag{11}$$

We can then rearrange this equation to obtain the following:

$$\rho_i^{n+1}-\rho_i^n=\Delta\rho=\frac{\delta t}{\delta x}(\rho_{i+1}^n-\rho_i^n)(A-B\rho_i^n) \tag{12}$$

5. Simple 1-Dimensional Case Study

Using Equation 12, we implemented our one-dimensional crowd flow model in Matlab using the following parameters:

1. The time step size, Δt, is 0.02 seconds
2. The x step size, Δx, 0.03 meters
3. The simulation is run over the span of 50 seconds
4. The simulation is looking at a walking path that is 35 meters in length
5. The constant $A = 1.4$m/s
6. The constant $B = 0.25\text{m}^2$/s

Note that we have selected our time and x step sizes specifically to ensure that our model is stable. The initial condition we applied was a Gaussian located at the center of the x domain of the model with a maximum height of 2. This is equivalent to dropping a bunch of people in the center of the street such that their density in the middle of the length of that street is 2 people/m. Therefore, our initial condition can be plotted as shown below:

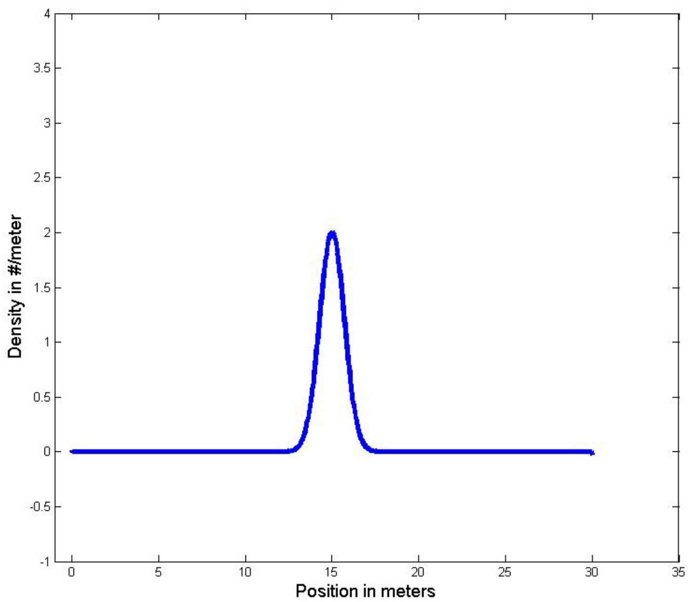

Figura 1. Plot of Initial Condition

When we turn on time, we obtain the following graph after some time has elapsed:

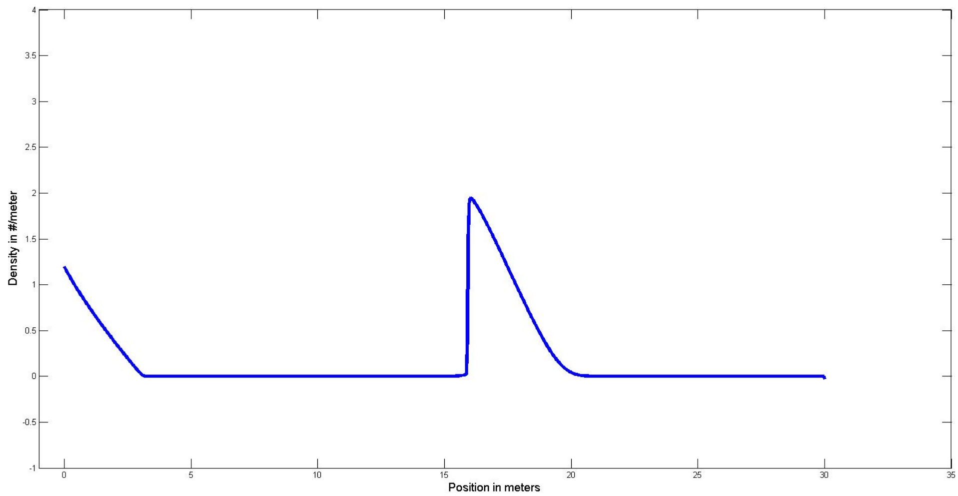

Figura 2. Plot of the density distribution across the x domain after some time t has passed

As can be seen in the above figure, as the Gaussian travels across the graph, it spreads out because we have used a speed function $f(\rho)$ that as been specifically designed such that pedestrians at lower density are traveling faster than those at higher density, causing the wave to change shape as it travels.The area under the Gaussian remains the same until the right edge touches the boundary, at which people begin to leave. Notice also that on the left side, people pile near the entrance as we enforce the in-flow rate. Density is high near the entrance, and drops to 0 as people spread out across the x domain.

6. Concluding Remarks

In short, our project is centered on the concept that for most relatively large crowds, pedestrians inside the crowd will want to approach there destination through a path that minimizes their travel time and high density situations that they might encounter. We take the mass conservation equation from fluid dynamics and apply our own formulas to create an equation that can model crowds through the use of concepts that treat crowds as waves or as discrete situations.

If one wishes to learn more about continuum-based crowd flow modeling, there are several good sources to use. To begin, one can start by reading the Hughes (2002) paper that fully describes the method of modeling crowd flow that we have presented in this handout. The papear also provides several case studies of how his method may be applied to 2-dimensional crowd flow modeling. Then, for a further refined model, one could refer to Huang et al. (2009) which refines the model proposed in Hughes (2002) and offers a more rigorous numerical solution method. A good problem to then try and solve is to begin by modeling crowd flow in the 1-dimensional case followed by the 2-dimensional case.

7. The Full PDE Model Development Process

7.1. Deriving the PDEs

As promised, for the interested reader who wishes to see a more rigorous derivation of the PDEs we used above, we will now discuss the derivation of the PDEs we used to model crowd flow more fully. Just as we began in Section 3, we begin with the simplest model form that makes the most sense would be some sort of mass conservation law like this one:

$$\frac{\delta\rho}{\delta t} + \frac{\delta}{\delta x}(\rho u) + \frac{\delta}{\delta y}(\rho v) = 0 \tag{13}$$

In essence, Equation 13 applies the concept of mass consevation but is really a statement of ”people conservaion”. Physically, Equation 13 states that the change in density over time is the result of a net flow of people into or out of an area.

Given that we want to solve for the density of a crowd, we need to know more about how to model the velocities, u and v, of the crowd. To do his, we make our first major assumption:

Assumption 1: The speed at which pedestrians walk is only dependent on the density of the pedestrians as well as the behavioral characteristics of the crowd.

Assuming the crowd is comprised of pedestrians all having the same goal, we can therefore express the velocity components of the movement of the crowd as:

$$u = f(\rho)\hat{\phi}_x \tag{14}$$

$$v = f(\rho)\hat{\phi}_y \tag{15}$$

In Equations 14 and 15, the ”behavioral characerisics of the crowd”are captured in the $\hat{\phi}_x$ and $\hat{\phi}_y$ terms. Those terms are commonly referred to as the direction cosines of the motion or the direction in which the pedestrians are walking.

At this point, we sill need an actual equation that captures the behavior of the crowd. To develop this equation, we make the following assumption:

Assumption 2: Pedestrians have a sense of task called potential that drives them to reach their goal or destination.

In essence, potential is a measure of how much a crowd wishes to get from their current position to their destination. Thus, given a crowd of pedestrians and applying assumption 2, one can imagine a series of lines of constant potential, where each line of constant potential represents all the points from which a pedesrian will have an equal desire to get from that point to their destination. An implication of this assumption is that a pedesrian will desire to reach their destination and therefore reduce their potential.

In other words, there should be no advantage for a pedesrian to move along the lines of potential. Therefore, the crowd will move in a direction that is perpendicular to lines of constant poential. Mathemaically, we can express this as:

$$\hat{\phi_x} = \frac{-\frac{\delta\phi}{\delta x}}{\sqrt{(\frac{\delta\phi}{\delta x})^2 + (\frac{\delta\phi}{\delta y})^2}} \tag{16}$$

$$\hat{\phi_y} = \frac{-\frac{\delta\phi}{\delta y}}{\sqrt{(\frac{\delta\phi}{\delta x})^2 + (\frac{\delta\phi}{\delta y})^2}} \tag{17}$$

where ϕ is the potential.

As can be seen in Equations 16 and 17, $\frac{\delta\phi}{\delta x}$ and $\frac{\delta\phi}{\delta y}$ are the gradient of potential which points perpendicular to the lines of constant potential. In this way, we have captured the behavior of the crowd to move perpendicular to lines of constant potential. The negative sign is present because the gradient points in the direcion of steepest ascent whereas crowds will move in the direction of steepest descent. The denominator in Equations 16 and 17 are then present because we want to obtain unit vectors.

At this point, the sources we read also recommend implementing one other equation to properly capture crowd flow behavior. That behavior is captured in the following assumption:

Assumption 3: Pedestrians would like to minimize their travel time, but would also like to avoid high density situations.

To capture this, Hughes (2002) introduces a term $g(\rho)$ that represents a sort of discomfort function. In essence, this discomfort function tries o capture the fact that pedestrians will adjust their behavior to minimize the product of their travel time and and avoid high density areas. Mathematically, this can be expressed as:

$$\frac{1}{\sqrt{(\frac{\delta\phi}{\delta x})^2 + (\frac{\delta\phi}{\delta y})^2}} = g(\rho)\sqrt{u^2 + v^2} \tag{18}$$

Now that we have Equations 13, 14, 15, 16 and 17, we can combine them together to obtain the first PDE that governs pedestrian flow:

$$-\frac{\delta\rho}{\delta t} + \frac{\delta}{\delta x}(\rho g(\rho) f^2(\rho)\frac{\delta\phi}{\delta x}) + \frac{\delta}{\delta y}(\rho g(\rho) f^2(\rho)\frac{\delta\phi}{\delta y}) = 0 \tag{19}$$

We also need one additional equation to relate the potential, ϕ to ρ and that equation is Equation 18. Since $f(\rho) = \sqrt{u^2 + v^2}$:

$$g(\rho) f(\rho) = \frac{1}{\sqrt{(\frac{\delta\phi}{\delta x})^2 + (\frac{\delta\phi}{\delta y})^2}} \tag{20}$$

7.2. Developing the Numerical Solution

Using Equations 19 and 20, we developed a numerical solution for the one dimensional case of crowd flow in Matlab. The following is a brief discussion of how that numerical model was developed.

Given that we are only studying the 1-dimensional case, we are essentially assuming that the crowd is invariant in the y direction and therefore all derivatives in y are equal to zero. Furthermore, according to Hughes (2002), $g(\rho)$ can be assumed to be equal to 1. As a result, Equation 19 can be re-written as:

$$-\frac{\delta\rho}{\delta t} + \frac{\delta}{\delta x}(\rho f^2(\rho)\frac{\delta\phi}{\delta x}) = 0 \tag{21}$$

Equally, Equation 20 can be re-written as:

$$f(\rho) = \frac{1}{\sqrt{(\frac{\delta\phi}{\delta x})^2}} = \frac{1}{|\frac{\delta\phi}{\delta t}|} \tag{22}$$

Finally, researchers such as Hughes (2002) and Huang et. al (2009) recommend starting with a linear model for the walking speed of the pedestrians:

$$f(\rho) = A - B\rho \tag{23}$$

where A is the free (i.e. when density is zero) walking speed of pedestrians and B controls the rate at which density affects the walking speed of pedestrains.

So, by applying Equations 22 and 23 to Equation 21, the following equation is obained:

$$\frac{\delta\rho}{\delta t} = \frac{\delta}{\delta x}(\rho(A - B\rho)) \tag{24}$$

$$\rho_t = (A\rho - B\rho^2)_x \tag{25}$$

$$\rho_t = (A - 2B\rho)\rho_x \tag{26}$$

Now that we have Equation 26, we can now solve the PDE numerically. If we use i to index our space unit and n to index our time unit, the following equation is obtained:

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} = \frac{\rho_i^n - \rho_{i-1}^n}{\delta x}(A - B\rho_i^n) \tag{27}$$

We can then rearrange this equation to obtain the following:

$$\rho_i^{n+1} - \rho_i^n = \Delta\rho = \frac{\delta t}{\delta x}(\rho_{i+1}^n - \rho_i^n)(A - B\rho_i^n) \tag{28}$$

7.2.1. Calculating the Stability Criterion

The final piece that one needs in order to implement a numerical method is to determine the criteria by which the numerical method will be stable or unstable. For the specific case of the crowd flow modeling PDEs that we have been working with, the stability criteria is a little more difficult to compute. However, we can compute the stability criteria for an easier, but related, case: the simple transport (i.e. mass conservation) equation.

Given the transport equation:

$$u_t = u_x \tag{29}$$

We can express this numerically as:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}(u_i^n + 1 - u_i^n) \tag{30}$$

Since we want u to be decreasing over time, then $\frac{\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$ needs to be less than one. If we propose that u_i^n is of the form:

$$u_i^n = e^{k\sqrt{-1}i} \tag{31}$$

where i is the space step, not the imaginary number.

Using Equation 31, we can re-express Equation 30 as:

$$u_{i+1}^n - u_i^n = (e^{k\sqrt{-1}} - 1)e^{k\sqrt{-1}i} \tag{32}$$

$$u_i^{n+1} = u_i^n(1 + \frac{\Delta t}{\Delta x}(e^{k\sqrt{-1}} - 1)) \tag{33}$$

We can therefore express the exponential in terms of sines and cosines:

$$u_i^{n+1} = u_i^n(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x}cos(k) + (\frac{\Delta t}{\Delta x}sin(k))^2) \tag{34}$$

For readibility, let us make the substitution $s = \frac{\Delta t}{\Delta x}$. Therefore, the above equation becomes:

$$u_i^{n+1} = u_i^n(1 - s + s * cos(k) + (s * sin(k))^2) \tag{35}$$

Looking at the above equation, we want to ensure the magnitude of coefficient of u_i^n is less than 1 but greater than zero so that the function does not grow:

$$0 < \sqrt{(1 - s + s * cos(k))^2 + (s * sin(k))^2} < 1 \tag{36}$$

Simplifying the above,

$$0 < 1 + (1 - cos(k))2s(s-1) < 1 \tag{37}$$

Breaking down the above equation, we know that $(1 - cos(k))$ will always be positive. Also, s will always be positive because our time and space steps will always be positive.

Thus, in order to satisfy the inequality, we know that the $(s - 1)$ term needs to be negative, implying:

$$0 < \frac{\Delta t}{\Delta x} < 1 \tag{38}$$

This is known as the stability criterion. Based on this inequality, during the implementation of a numerical solution, the time step in time must be smaller than the time step in x in order for the numerical solution to be stable.